Loop diagrams in space with SU(2) fuzziness

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Abstract

The structure of loop corrections is examined in a scalar field theory on a three dimensional space whose spatial coordinates are noncommutative and satisfy SU(2) Lie algebra. In particular, the 2- and 4-point functions in ϕ^4 scalar theory are calculated at the 1-loop order. The theory is UV-finite as the momentum space is compact. It is shown that the non-planar corrections are proportional to a one dimensional δ-function, rather than a three dimensional one, so that in transition rates only the planar corrections contribute.

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1 Introduction

In recent years there has been considerable interest in quantum field theories on noncommutative spaces. This was to a large extent motivated by the observation that this kind of field theories arise in the zero-slope limit of the open string theory in the presence of a constant B-field background [1–4]. In this case the coordinates satisfy the canonical relation

$$[\widehat{x}_\mu, \widehat{x}_\nu] = \mathrm{i}\, \theta_{\mu\nu}\, \mathbf{1}, \tag{1}$$

in which θ is an antisymmetric constant tensor and $\mathbf{1}$ represents the unit operator. The theoretical and phenomenological implications of such noncommutative coordinates have been extensively studied; see [5].

One direction to extend studies on noncommutative spaces is to consider spaces where the commutators of the coordinates are not constants. Examples of this kind are the noncommutative cylinder and the q-deformed plane [6], the so-called κ-Poincaré algebra [7–10], and linear noncommutativity of the Lie algebra type [11]. In the latter it is supposed that the dimensionless spatial positions operators satisfy the commutation relations of a Lie algebra [11]:

$$[\widehat{x}_a, \widehat{x}_b] = f^c{}_{a\,b}\, \widehat{x}_c, \tag{2}$$

where $f^c{}_{a\,b}$'s are structure constants of a Lie algebra. One example of this kind is the algebra SO(3), or SU(2). A special case of this is the so called fuzzy sphere [12, 13], where an irreducible representation of the position operators is used which makes the Casimir of the algebra, $(\widehat{x}_1)^2 + (\widehat{x}_2)^2 + (\widehat{x}_3)^2$, a multiple of the identity operator (a constant, hence the name sphere). One can consider the square root of this Casimir as the radius of the fuzzy sphere. This is, however, a noncommutative version of a two-dimensional space (sphere).

In [14, 15] a model was introduced in which the representation was not restricted to an irreducible one, instead the whole group was employed. In particular the regular representation of the group, which contains all representations, was considered. As a consequence in such models one is dealing with the whole space, rather than a sub-space, like the case of fuzzy sphere as a 2-dimensional surface. In [14] the basic ingredients for calculus on a linear fuzzy space, as well as the basic notions for a field theory on such a space, were introduced. In [15] the basic elements for calculating the matrix elements corresponding to transition between initial and final states were discussed. There the contributions of lowest order (tree level) perturbative expansion of amplitudes were presented for a self-interacting scalar field theory. The models based on the regular representations of SU(2) was treated in more detail, giving the explicit form of the tools and notions introduced in their general forms [14, 15].

As mentioned in [14, 15], one of the features of models based on linear fuzziness of Lie algebra type is that these theories are free from any ultraviolet divergences if the corresponding Lie group is compact. In fact one can consider the momenta as the coordinates of the group, so that the space of the corresponding momenta is compact iff the group is compact. One important

implication of the elimination of the ultraviolet divergences would be that there will be no place for the so called UV/IR mixing effect [16], which is known to be a common feature of the models based on canonical noncommutativity, the algebra (1).

The purpose of the present work is to examine the structure of the field theory amplitudes at loop order. Here we consider a scalar field theory with ϕ^4 interaction. In particular we consider one-loop corrections to 2- and 4-point functions in this theory. The field theory on a 2+1 spacetime whose coordinates satisfy the Lie algebra of SO(2,1) was studied in [17]. Due to non-compactness of the group in this case, the UV-divergences are present at loop level [17].

The scheme of the rest of this paper is the following. In section 2, a brief review is given on basic elements of a field theory on a noncommutative space of SU(2) algebra type. In sections 3 and 4 the calculation of 2- and 4-point functions are presented, respectively. Section 5 is devoted to concluding remarks; in particular, it is discussed how only the planar sector of the loop corrections contribute to the amplitudes.

2 Field theory on space with SU(2) fuzziness

In [14,15] a model was investigated in a 3+1 dimensional space-time the dimensionless spatial position operators of which are generators of a *regular* representation of the SU(2) algebra, that is

$$[\widehat{x}_a, \widehat{x}_b] = \epsilon^c{}_{a\,b}\,\widehat{x}_c. \tag{3}$$

As it was discussed in [14], one can use the group algebra as the analogue of functions defined on ordinary space, with group elements $U = \exp(\ell\, k^a\, \widehat{x}_a)$ as the analogues of $\exp(\mathrm{i}\,\mathbf{k}\cdot\mathbf{x})$, which are a basis for the functions defined on the space. In both cases $\mathbf{k}$ is an ordinary vector with $\mathbf{k} = (k^1, k^2, k^3)$. That is the components of $\mathbf{k}$ are commuting numbers. In the case of noncommutative space, ℓ is a length parameter, and the vector $\mathbf{k}$ is restricted to a ball of radius $(2\,\pi/\ell)$, with all points of the boundary identified to a single point. The manifold of $\mathbf{k}$ is in fact a 3-sphere. $\mathbf{k}$ can be thought of as the momentum of a particle. The left-right-invariant Haar measure is

$$\mathrm{d}U = \frac{\sin^2(\ell\,k/2)}{(\ell\,k/2)^2}\,\frac{\mathrm{d}^3k}{(2\,\pi)^3}, \tag{4}$$

where $k := |\mathbf{k}|$. The integration region for the coordinates is $k \leq 2\,\pi/\ell$. We mention that near the origin ($k \ll \ell^{-1}$) the measure is simply $\mathrm{d}^3k/(2\,\pi)^3$, as it should be. The action of a scalar model with quartic interaction in Fourier space of spatial directions is given by

$$S = \int \mathrm{d}t \Bigg\{ \frac{1}{2} \int \mathrm{d}U_1\,\mathrm{d}U_2\ \Big[\dot{\phi}(U_1)\,\dot{\phi}(U_2) + \phi(U_1)\,O(U_2)\,\phi(U_2)\Big]\ \delta(U_1\,U_2)$$

$$-\frac{g}{4!}\int\Big[\prod_{j=1}^{4}\mathrm{d}U_j\Big]\ \phi(U_1)\,\phi(U_2)\,\phi(U_3)\,\phi(U_4)\,\delta(U_1\,U_2\,U_3\,U_4)\Bigg\}, \quad (5)$$

in which $\dot{\phi}$ is the time derivative of ϕ. In the above,

$$O(U) = c\,\chi_\lambda(U + U^{-1} - 2\,\mathbf{1}) - m^2, \quad (6)$$

where c and m are constants, and χ_λ is the character in the representation λ. It is shown that by a proper choice of constant c, near the origin $O(U) \approx -k^2 - m^2$, as it is the case in the ordinary space. The δ-distribution appearing above is simply defined through

$$\int \mathrm{d}U\ \delta(U)\,f(U) := f(\mathbf{1}), \quad (7)$$

where $\mathbf{1}$ is the identity element of the group. It is easy to see that this delta distribution is invariant under similarity transformations, as well as inversion of the argument:

$$\begin{aligned}\delta(V\,U\,V^{-1}) &= \delta(U),\\ \delta(U^{-1}) &= \delta(U).\end{aligned} \quad (8)$$

The first relation shows that if the argument of the delta is a product of group elements, then any cyclic permutation of these elements leaves the delta unchanged. It is also seen that near the origin ($k \ll \ell^{-1}$),

$$\delta(U_1\ \cdots\ U_l) \approx (2\,\pi)^3\,\delta^3(\mathbf{k}_1 + \cdots + \mathbf{k}_l), \quad (9)$$

which ensures an approximate momentum conservation. The exact conservation law, however, is that at each vertex the product of incoming group elements should be unity. For the case of a 3-leg vertex, one can write this condition as

$$\exp(\ell\,k_1^a\,\widehat{x}_a)\ \exp(\ell\,k_2^a\,\widehat{x}_a)\ \exp(\ell\,k_3^a\,\widehat{x}_a) = \mathbf{1}. \quad (10)$$

It is convenient to define

$$\exp(\ell\,k_1^a\,\widehat{x}_a)\ \exp(\ell\,k_2^a\,\widehat{x}_a) =: \exp[\ell\,\gamma^a(\mathbf{k}_1,\mathbf{k}_2)\,\widehat{x}_a], \quad (11)$$

where the function $\boldsymbol{\gamma}$ can be shown to enjoy the properties

$$\boldsymbol{\gamma}[\mathbf{k}_1,\boldsymbol{\gamma}(\mathbf{k}_2,\mathbf{k}_3)] = \boldsymbol{\gamma}[\boldsymbol{\gamma}(\mathbf{k}_1,\mathbf{k}_2),\mathbf{k}_3], \quad (12)$$
$$\boldsymbol{\gamma}(-\mathbf{k}_1,-\mathbf{k}_2) = -\,\boldsymbol{\gamma}(\mathbf{k}_2,\mathbf{k}_1), \quad (13)$$
$$\boldsymbol{\gamma}(\mathbf{k},-\mathbf{k}) = 0. \quad (14)$$

So that (10) can be expressed by one of the three equivalent forms

$$\begin{aligned}\mathbf{k}_3 &= -\,\boldsymbol{\gamma}(\mathbf{k}_1,\mathbf{k}_2),\\ \mathbf{k}_2 &= -\,\boldsymbol{\gamma}(\mathbf{k}_3,\mathbf{k}_1),\\ \mathbf{k}_1 &= -\,\boldsymbol{\gamma}(\mathbf{k}_2,\mathbf{k}_3).\end{aligned} \quad (15)$$

The explicit form of $\boldsymbol{\gamma}(\mathbf{k}_1, \mathbf{k}_2)$ is obtained from

$$\begin{aligned}
\cos\frac{\ell\,\gamma}{2} =&\ \cos\frac{\ell\,k_1}{2}\,\cos\frac{\ell\,k_2}{2} - \hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2\,\sin\frac{\ell\,k_1}{2}\,\sin\frac{\ell\,k_2}{2},\\
\hat{\boldsymbol{\gamma}}\,\sin\frac{\ell\,\gamma}{2} =&\ \hat{\mathbf{k}}_1\times\hat{\mathbf{k}}_2\,\sin\frac{\ell\,k_1}{2}\,\sin\frac{\ell\,k_2}{2}\\
&+\hat{\mathbf{k}}_1\,\sin\frac{\ell\,k_1}{2}\,\cos\frac{\ell\,k_2}{2}+\hat{\mathbf{k}}_2\,\sin\frac{\ell\,k_2}{2}\,\cos\frac{\ell\,k_1}{2}.
\end{aligned} \tag{16}$$

It is easy to see that in the limit $\ell \to 0$, $\boldsymbol{\gamma}$ tends to $\mathbf{k}_1 + \mathbf{k}_2$, as expected.

For field theoretical purposes it is convenient to have the action (5) in the whole (space and time) Fourier space:

$$\begin{aligned}
S =&\frac{1}{2}\int\frac{\mathrm{d}\omega_1\,\mathrm{d}U_1}{2\,\pi}\,\frac{\mathrm{d}\omega_2\,\mathrm{d}U_2}{2\,\pi}\,\left[2\,\pi\,\delta(\omega_1+\omega_2)\,\delta(U_1\,U_2)\right]\\
&\times\left[-\omega_1\,\omega_2\,\check{\phi}(U_1,\omega_1)\,\check{\phi}(U_2,\omega_2)+\check{\phi}(U_1,\omega_1)\,O(U_2)\,\check{\phi}(U_2,\omega_2)\right]\\
&-\frac{g}{4!}\int\left[\prod_{j=1}^{4}\frac{\mathrm{d}\omega_j\,\mathrm{d}U_j}{2\,\pi}\,\check{\phi}(U_j,\omega_j)\right]\,\left[2\,\pi\,\delta(\omega_1+\cdots+\omega_4)\,\delta(U_1\cdots U_4)\right],
\end{aligned} \tag{17}$$

in which $\check{\phi}(U, \omega)$ is the Fourier component. The first two terms represent a free action, with the propagator

$$\check{\Delta}(\omega, U) := \frac{\mathrm{i}\,\hbar}{\omega^2 + O(U)}. \tag{18}$$

Putting the denominator of this propagator equal to zero gives the relation between ω and U for free particles (the mass-shell condition). The third term contains interactions. Any Feynman graph would consist of propagators, and 4-line vertices to which one assigns the fundamental vertex

$$\mathcal{V}_{[1234]} := \frac{g}{\mathrm{i}\,\hbar\,4!}\,2\,\pi\,\delta(\omega_1+\cdots+\omega_4)\,\sum_{\Pi}\delta(U_{\Pi(1)}\cdots U_{\Pi(4)}), \tag{19}$$

where the summation runs over all permutations. In practice, due to cyclic symmetry of δ's arguments mentioned earlier, permutations which are different up to a cyclic change just come in sum with a proper weight, so we have

$$\begin{aligned}
\mathcal{V}_{[1234]} =&\frac{g}{\mathrm{i}\,\hbar\,6}\,2\,\pi\,\delta(\omega_1+\cdots+\omega_4)\,\Big[\delta(U_1U_2U_3U_4)+\delta(U_1U_2U_4U_3)\\
&+\delta(U_1U_3U_2U_4)+\delta(U_1U_3U_4U_2)+\delta(U_1U_4U_2U_3)+\delta(U_1U_4U_3U_2)\Big].
\end{aligned} \tag{20}$$

Also, for any internal line there is an integration over U and ω, with the measure $\mathrm{d}\omega\,\mathrm{d}U/(2\,\pi)$. As the group is assumed to be compact, the integration over the group is integration over a compact volume.

It is worth to mention a crucial difference between the way that δ-functions appear in our model and in models defined on ordinary spaces. Here, as mentioned above, each possible ordering of legs of a vertex comes with a different

δ, except the cases that two orderings are different up to a cyclic permutation. This is in contrast to models on ordinary space, in which all possible orderings have the common factor of one single $\delta\big(\sum \mathbf{k}_i\big)$, representing the momentum conservation in that vertex. Similar to above observation about the appearance of δ-functions has been made in theories defined on κ-deformed spaces, pointed in the Introduction. In these theories, the ordinary summation of momenta in each vertex is replaced with a new rule of summation, occasionally called as doted-sum ($\dot{+}$) [9]. This new sum, in contrast to the ordinary sum, is non-Abelian, and as a consequence, the δ's coming with each possible ordering of legs are different [9, 10].

Once given by the Feynman rules one can calculate the loop corrections. In following we choose $\lambda = \frac{1}{2}$ in (6), so that the propagator has the explicit form

$$\check{\Delta}(\omega, \mathbf{k}) = \frac{\mathrm{i}\,\hbar}{\omega^2 - \dfrac{16}{\ell^2}\sin^2\dfrac{\ell\,k}{4} - m^2}. \tag{21}$$

3 1-loop correction of the 2-point function

The 2-point function has two external legs, one incoming $(\omega_1, \mathbf{k}_1)$, the other outgoing $(-\omega_2, -\mathbf{k}_2)$. The 1-loop correction is simply the fundamental vertex-function (20), contracted on legs 3 and 4, with proper symmetry factors:

$$\begin{aligned}\Gamma^{(2)}_{1-\mathrm{loop}} =& \frac{1}{2}\int \mathrm{d}U_3\,\mathrm{d}U_4\;\delta(U_3\,U_4)\int \frac{\mathrm{d}\omega_3}{2\pi}\,\frac{\mathrm{d}\omega_4}{2\pi}\;2\,\pi\,\delta(\omega_3+\omega_4)\,\mathcal{V}_{[1234]},\\ =& \frac{g}{\mathrm{i}\,\hbar\,12}\,2\,\pi\,\delta(\omega_1-\omega_2)\int \mathrm{d}U\int \frac{\mathrm{d}\omega}{2\pi}\;\frac{\mathrm{i}\,\hbar}{\omega^2+O(U)}\\ &\times\left[4\,\delta(U_1\,U_2^{-1}) + 2\,\delta(U_1\,U\,U_2^{-1}\,U^{-1})\right],\end{aligned} \tag{22}$$

where the integrations on ω_4 and U_4 have been performed, and ω_3 and U_3 have been denoted by ω and U, respectively. Also use has been made of the facts that $\mathrm{d}U$ and $O(U)$ are invariant under the inversion of U. In the above expression, one recognizes two parts: the so called planar part, in which the delta contains no contribution from the loop momentum U; and the so called nonplanar part, in which the delta does contain loop momentum. Using the usual prescription $\omega^2 \to (\omega^2 + \mathrm{i}\,\varepsilon)$, the integration over omega is performed. For the planar part, one obtains

$$\begin{aligned}\Gamma^{(2)\mathrm{planar}}_{1-\mathrm{loop}} &= -\frac{\mathrm{i}\,g}{6}\,2\,\pi\,\delta(\omega_1-\omega_2)\,\delta(U_1\,U_2^{-1})\int \frac{\mathrm{d}U}{\sqrt{-O(U)}}\\ &= -\frac{\mathrm{i}\,8\,\pi\,g}{3\,\ell^2}\,2\,\pi\,\delta(\omega_1-\omega_2)\,\delta(U_1\,U_2^{-1})\int_0^{2\,\pi/\ell}\frac{\mathrm{d}k}{(2\,\pi)^3}\,\frac{\sin^2(\ell\,k/2)}{\sqrt{16\sin^2(\ell\,k/4)+m^2}}.\end{aligned} \tag{23}$$

The remaining integral can be expressed in terms of a hypergeometric function of ${}_2F_1(a,b,c;z)$ type. It is seen that the above expression is finite, as it was to be. The delta distribution of group elements can also be written in the form

$$\delta(U_1\,U_2^{-1}) = \frac{(2\,\pi)^3\,(\ell\,k_2/2)^2}{\sin^2(\ell\,k_2/2)}\,\delta^3(\mathbf{k}_1-\mathbf{k}_2). \tag{24}$$

It is seen that planar contribution conserves momentum.

For the non-planar contribution, one has

$$\Gamma^{(2)\text{non--planar}}_{1-\text{loop}} = -\frac{\mathrm{i}\,g}{12}\,2\,\pi\,\delta(\omega_1-\omega_2)\int \mathrm{d}U\,\frac{\delta(U_1\,U\,U_2^{-1}\,U^{-1})}{\sqrt{-O(U)}}. \tag{25}$$

It is convenient to define $\mathbf{k}_2'$ through

$$\begin{aligned} U(\mathbf{k}_2') &:= U(\mathbf{k})\,U_2\,U^{-1}(\mathbf{k}),\\ \mathbf{k}_2' &= \mathbf{k}_2\,\cos(\ell\,k) + \hat{\mathbf{k}}\,(\hat{\mathbf{k}}\cdot\mathbf{k}_2)\,[1-\cos(\ell\,k)] + \mathbf{k}_2\times\hat{\mathbf{k}}\,\sin(\ell\,k). \end{aligned} \tag{26}$$

In fact $\mathbf{k}_2'$ is nothing but $\mathbf{k}_2$ rotated by the angle $\ell\,k$ around $\hat{\mathbf{k}}$. So,

$$k_2' = k_2, \tag{27}$$

and

$$\begin{aligned} \delta(U_1\,U\,U_2^{-1}\,U^{-1}) &= \delta(U_1\,U_2'^{-1}),\\ &= \frac{(2\,\pi)^3(\ell\,k_2/2)^2}{\sin^2(\ell\,k_2/2)}\,\delta^3(\mathbf{k}_1-\mathbf{k}_2'). \end{aligned} \tag{28}$$

It would be helpful to express this delta in terms of spherical coordinates:

$$\delta^3(\mathbf{k}_1-\mathbf{k}_2') = \frac{1}{k_1^2}\,\delta(k_1-k_2)\,\delta(\cos\theta_1-\cos\theta_2')\,\delta(\phi_1-\phi_2'). \tag{29}$$

Without loss of generality, one can put the 3rd direction on $\mathbf{k}_2$. So,

$$\begin{aligned} \mathbf{k}_2 &= k_2\,\hat{\mathbf{z}},\\ \cos\theta_2' &= \cos(\ell\,k) + \cos^2\theta\,[1-\cos(\ell\,k)]\\ \phi_2' &= \phi - \tan^{-1}\left\{\frac{\sin(\ell\,k)}{\cos\theta\,[\cos(\ell\,k)-1]}\right\}, \end{aligned} \tag{30}$$

where (k,θ,ϕ) are the spherical coordinates of $\mathbf{k}$. Since $O(U)$ is independent of θ and ϕ, one can use

$$\begin{aligned} I_1 &:= \int_{-1}^{1}\mathrm{d}(\cos\theta)\;\delta\left\{\cos\theta_1-\cos(\ell\,k)-\cos^2\theta\,[1-\cos(\ell\,k)]\right\},\\ &= \begin{cases} \{[1-\cos(\ell\,k)]\,[\cos\theta_1-\cos(\ell\,k)]\}^{-1/2}, & \theta_1\le\ell\,k\le 2\,\pi-\theta_1\\ 0, & \text{otherwise} \end{cases}\\ I_2 &:= \int \mathrm{d}\phi\;\delta(\phi_1-\phi_2') = 1. \end{aligned} \tag{31}$$

One then arrives at

$$
\begin{aligned}
\Gamma^{(2)\text{non–planar}}_{1-\text{loop}} &= -\frac{\mathrm{i}\, g}{12}\, 2\,\pi\,\delta(\omega_1-\omega_2)\,\delta(k_1-k_2)\,\frac{(2\,\pi)^3\,\ell^2/4}{\sin^2(\ell\, k_2/2)} \\
&\quad\times \int_{\theta_1/\ell}^{(2\,\pi-\theta_1)/\ell} \frac{\mathrm{d}k}{(2\,\pi)^3}\, \frac{\sin^2(\ell, k/2)}{\sqrt{16\sin^2(\ell\, k/4)+m^2}} \\
&\quad\times \frac{1}{\sqrt{[1-\cos(\ell\, k)]\,[\cos\theta_1-\cos(\ell\, k)]}}.
\end{aligned}
\tag{32}
$$

Putting $\cos^{-1}(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)$ instead of θ_1, an expression is obtained that has no particular choice of coordinates. The above expression is finite, hence no appearance of UV-divergences in either planar or non-planar diagrams. However, in the above expression one has a one-dimensional delta $\delta(k_1-k_2)$, instead of the three dimensional delta $\delta^3(\mathbf{k}_1-\mathbf{k}_2)$ appearing in the planar part. So the non-planar part does not leave the momentum vector but only its length conserved. So it is possible that the direction of the momentum of a self-interacting particle changes. This is no surprise, as in this theory momentum conservation is just an approximate law. Similar observations on the change of momentum through non-planar loop corrections to 2-point functions have been reported in κ-Poincaré theories [9, 10]

4 1-loop correction of the 4-point function

The 4-point function has four external legs: two incomings $(\omega_1,\mathbf{k}_1)$ and $(\omega_2,\mathbf{k}_2)$, and two outgoings $(-\omega_3,-\mathbf{k}_3)$ and $(-\omega_4,-\mathbf{k}_4)$. The contributions come from three channels, the so-called s-, t-, and u-channels. Here only the s-channel is investigated, as the two others can be obtained similarly. One has

$$
\begin{aligned}
\Gamma^{(4)[\mathrm{s}]}_{1-\text{loop}} &= \frac{1}{2}\int \mathrm{d}U\mathrm{d}U' \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}\omega'}{2\pi} \\
&\quad\times \frac{\mathrm{i}\,\hbar}{\omega^2+O(U)}\,\frac{\mathrm{i}\,\hbar}{\omega'^2+O(U')}\,\mathcal{V}_{[12U^{-1}U'^{-1}]}\,\mathcal{V}_{[UU',-3,-4]}.
\end{aligned}
\tag{33}
$$

Using (20), and performing the ω'-integration, one would get

$$
\begin{aligned}
\Gamma^{(4)[\mathrm{s}]}_{1-\text{loop}} &= \frac{1}{2}\left(\frac{g}{6\,\mathrm{i}\,\hbar}\right)^2 2\pi\,\delta(\omega_1+\omega_2-\omega_3-\omega_4)\int \mathrm{d}U\mathrm{d}U' \int\frac{\mathrm{d}\omega}{2\,\pi} \\
&\quad\times \frac{\mathrm{i}\,\hbar}{\omega^2+O(U)}\,\frac{\mathrm{i}\,\hbar}{(\omega_{\mathrm{s}}-\omega)^2+O(U')} \\
&\quad\times [\delta(U_1\,U_2\,U^{-1}\,U'^{-1})+\delta(U_1\,U_2\,U'^{-1}\,U^{-1})+\delta(U_1\,U^{-1}\,U_2\,U'^{-1}) \\
&\quad+\delta(U_1\,U'^{-1}\,U_2\,U^{-1})+\delta(U_2\,U_1\,U^{-1}\,U'^{-1})+\delta(U_2\,U_1\,U'^{-1}\,U^{-1})] \\
&\quad\times [\delta(U_3^{-1}\,U_4^{-1}\,U\,U')+\delta(U_3^{-1}\,U_4^{-1}\,U'\,U)+\delta(U_3^{-1}\,U\,U_4^{-1}\,U') \\
&\quad+\delta(U_3^{-1}\,U'\,U_4^{-1}\,U)+\delta(U_4^{-1}\,U_3^{-1}\,U\,U')+\delta(U_4^{-1}\,U_3^{-1}\,U'\,U)],
\end{aligned}
\tag{34}
$$

in which

$$\omega_{\rm s} := \omega_1 + \omega_2. \tag{35}$$

It would be convenient to define the followings

$$\begin{aligned} U'_a &:= U\, U_a\, U^{-1}, \\ U''_a &:= U^{-1}\, U_a\, U. \end{aligned} \tag{36}$$

Performing the U'-integration, and the change $\omega \to \omega - \omega_{\rm s}$, one arrives at

$$\begin{aligned} \Gamma^{(4)[{\rm s}]}_{1-{\rm loop}} =& \frac{1}{2}\, \left(\frac{g}{6}\right)^2 2\pi\, \delta(\omega_1+\omega_2-\omega_3-\omega_4) \int {\rm d}U \int \frac{{\rm d}\omega}{2\,\pi}\, \frac{1}{(\omega-\omega_{\rm s})^2 + O(U)} \\ & \times \left\{ \frac{1}{\omega^2 + O(U_1\, U_2\, U^{-1})} \, [2\, \delta(U_1\, U_2\, U_3^{-1}\, U_4^{-1}) + 2\, \delta(U_1\, U_2\, U_4^{-1}\, U_3^{-1}) \right. \\ & + \delta(U_1\, U_2\, U_3''^{-1}\, U_4^{-1}) + \delta(U_1\, U_2\, U_4''^{-1}\, U_3^{-1}) + \delta(U_1\, U_2\, U_3^{-1}\, U_4'^{-1}) \\ & + \delta(U_1\, U_2\, U_4^{-1}\, U_3'^{-1}) + \delta(U_1\, U_3^{-1}\, U_4^{-1}\, U_2') + \delta(U_1''\, U_3^{-1}\, U_4^{-1}\, U_2) \\ & + \delta(U_1\, U_4^{-1}\, U_3^{-1}\, U_2') + \delta(U_1''\, U_4^{-1}\, U_3^{-1}\, U_2) \\ & + \delta(U_1'\, U_2'\, U_4^{-1}\, U_3^{-1}) + \delta(U_1''\, U_2''\, U_3^{-1}\, U_4^{-1}) + \delta(U_1''\, U_2''\, U_4^{-1}\, U_3^{-1}) \\ & + \delta(U_1\, U_3^{-1}\, U_4'^{-1}\, U_2') + \delta(U_1\, U_4^{-1}\, U_3'^{-1}\, U_2') + \delta(U_1'\, U_2'\, U_3^{-1}\, U_4^{-1})] \\ & + \frac{1}{\omega^2 + O(U_2\, U_1\, U^{-1})} \, [2\, \delta(U_2\, U_1\, U_3^{-1}\, U_4^{-1}) + 2\, \delta(U_2\, U_1\, U_4^{-1}\, U_3^{-1}) \\ & + \delta(U_1'\, U_2\, U_3^{-1}\, U_4^{-1}) + \delta(U_1\, U_2''\, U_3^{-1}\, U_4^{-1}) + \delta(U_1'\, U_2\, U_4^{-1}\, U_3^{-1}) \\ & + \delta(U_1\, U_2''\, U_4^{-1}\, U_3^{-1}) + \delta(U_2\, U_1\, U_3''^{-1}\, U_4^{-1}) + \delta(U_2\, U_1\, U_4''^{-1}\, U_3^{-1}) \\ & + \delta(U_2\, U_1\, U_3^{-1}\, U_4'^{-1}) + \delta(U_2\, U_1\, U_4^{-1}\, U_3'^{-1}) \\ & + \delta(U_2''\, U_1''\, U_4^{-1}\, U_3^{-1}) + \delta(U_4'^{-1}\, U_1'\, U_2\, U_3^{-1}) + \delta(U_3'^{-1}\, U_1\, U_2\, U_4^{-1}) \\ & \left. + \delta(U_2'\, U_1'\, U_3^{-1}\, U_4^{-1}) + \delta(U_2'\, U_1'\, U_4^{-1}\, U_3^{-1}) + \delta(U_2''\, U_1''\, U_3^{-1}\, U_4^{-1})] \right\}, \end{aligned} \tag{37}$$

where use has been made of

$$O(ABC) = O(CAB). \tag{38}$$

In the above expression, the delta's come in two way: those without the loop variable, which correspond to the planar part; and those containing the loop variable (those which contain $'$ or $''$), which correspond to the non-planar part.

In the planar part, the delta's are simply brought out of the integral. So,

$$
\begin{aligned}
\Gamma^{(4)[\mathrm{s}]\mathrm{planar}}_{1-\mathrm{loop}} = \frac{g^2}{72\pi}\,\delta(\omega_\mathrm{s}-\omega_3-\omega_4) \Bigg\{ & [\delta(U_1\,U_2\,U_3^{-1}\,U_4^{-1}) + \delta(U_1\,U_2\,U_4^{-1}\,U_3^{-1})] \\
& \times \int \mathrm{d}U \int \frac{\mathrm{d}\omega}{[(\omega-\omega_\mathrm{s})^2+O(U)]\,[\omega^2+O(U_1\,U_2\,U^{-1})]} \\
& + [\delta(U_2\,U_1\,U_3^{-1}\,U_4^{-1}) + \delta(U_2\,U_1\,U_4^{-1}\,U_3^{-1})] \\
& \times \int \mathrm{d}U \int \frac{\mathrm{d}\omega}{[(\omega-\omega_\mathrm{s})^2+O(U)]\,[\omega^2+O(U_2\,U_1\,U^{-1})]} \Bigg\}. \qquad (39)
\end{aligned}
$$

Again the contribution of the planar part is proportional to three-dimensional delta's.

One can proceed to bring the above expressions in more simple forms, though in this case the integrand does not just depend on the length of momentum. As an easy example, one can consider the case where the reactions takes place in the so called center of mass frame:

$$
U_2 = U_1^{-1}. \qquad (40)
$$

One then arrives at

$$
\begin{aligned}
\Gamma^{(4)[\mathrm{s}]\mathrm{planar}}_{1-\mathrm{loop}} = & \frac{g^2}{18\pi}\,\delta(\omega_\mathrm{s}-\omega_3-\omega_4)\,\delta(U_3 U_4) \\
& \times \int \mathrm{d}U \int \frac{\mathrm{d}\omega}{[(\omega-\omega_\mathrm{s})^2+O(U)]\,[\omega^2+O(U)]}. \qquad (41)
\end{aligned}
$$

For the non-planar contribution, as it was the case with the two-point function, one cannot factor out a three-dimensional delta, as the loop variable is inside the argument of the delta's. It is again obvious that both the planar and non-planar contributions are finite.

5 Concluding remarks

The structure of loop corrections of a self-interacting field theory on a three dimensional space whose spatial coordinates are noncommutative and satisfy SU(2) Lie algebra was examined. The examples of 1-loop 2- and 4-point functions were treated in more detail as examples of loop corrections in such models. As the momentum space of such models are compact, the theory is free from UV divergences.

In the case of the 2-point function, while the planar part leaves the momentum conserved, the non-planar only leaves the length of the momentum conserved. In the case of the 4-point function, the momentum is conserved neither by the planar part nor by the non-planar part. Similar observations have already been done in the case of the κ-Poincaré case [9].

One notable feature of the model is about the ways that the δ-functions appear in planar and non-planar sectors of the theory. The planar contribution comes with a three dimensional δ, representing certain combinations of the external momentums of the diagrams. In the non-planar case, less-dimensional delta's remain. As a consequence, the n-point function can always come schematically as below

$$\Gamma^{(n)} = 2\,\pi\,\delta\left(\sum_{\text{i \& f}}\omega\right)\,\left[\sum_{\lambda}\Gamma_{\lambda}^{(n)\,\text{planar}}\delta^{3}(\mathbf{v}_{\lambda}) + \sum_{\mu}\Gamma_{\mu}^{(n)\,\text{non-planar}}\delta^{\alpha_{\mu}}(\mathbf{v}_{\mu})\right], \quad (42)$$

where λ and μ denote different orderings of external legs. The vectors $\mathbf{v}_\lambda$ and $\mathbf{v}_\mu$ are certain combinations of external momenta. Finally, the numbers α_μ are less than 3, meaning that the delta arising from the non-planar part is less than three-dimensional.

Observables (like cross sections and decay rates) are proportional to the square of $\Gamma^{(n)}$, in which there are terms proportional to $[\delta^{\alpha}(\mathbf{v})]\,[\delta^{\alpha'}(\mathbf{v}')]$, where α or α' is equal to 3 for contributions from the planar sector, and less than 3 for contributions from the non-planar sector. It is seen that in such products, there arise terms proportional to $\delta^{\beta}(\mathbf{0})$, such that $\beta = 3$ iff $\mathbf{v} = \mathbf{v}'$ and $\alpha = \alpha' = 3$, and $\beta < 3$ otherwise. So $\delta^3(\mathbf{0})$ arises only in the product of similar terms in the planar sector. As it was discussed in [15], to obtain transition rates these terms should be multiplied by other factors including powers of the volume of the space, which tends to $\delta^3(\mathbf{0})$ in the infinite volume limit. These factors cancel one $\delta^3(\mathbf{0})$ from the square, so that terms containing one $\delta^3(\mathbf{0})$ give finite contributions. Other terms vanish, and as terms containing one $\delta^3(\mathbf{0})$ arise only from the planar parts, the non-planar corrections do not contribute in the probability of any transition rate. The only exception is for the 2-point function, where the planar part has no contribution to the self-scattering (spontaneous momentum change) and the scattering is totally due to the non-planar part.

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